

Coarsifications of the module isomorphism relation

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Abstract

In this paper new equivalence relations on the category $Mod(A)$ for any associative algebra A and several related results are given. The new equivalence relations are defined using restrictions to subalgebras and the action of algebra automorphisms on modules.

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1 Introduction

The classic representation/module theory of associative algebras studies the module categories over algebras with their algebra structures having been fixed once and for all. Modules are usually studied up to isomorphism and isomorphism classes are orbits of the set of objects of $Mod(A)$ under the conjugation action of the general linear group of the underlying vector space of the module. In this paper we introduce new equivalence relations in the module category which are strictly coarser than the isomorphism relation. This is motivated by a desire to make progress in the classification of indecomposable modules in the difficult cases such as the algebras of wild representation type.

There are two main ideas. The first idea is to consider restrictions of a module to all subalgebras and collect the information into a single structure. This idea is not entirely new since restrictions to subalgebras of finite representation type have been studied for many years. The second idea is based on an additional action on the category of modules - the

action of the automorphism group of the algebra. The orbits of this action define an equivalence relation which is coarser than the standard isomorphism equivalence relation. This equivalence relation allows us to study representations of the algebra up to algebra automorphisms. The described new equivalences relations can contribute to the understanding of classic module categories in the tame and wild cases, in particular, some parameters of multiparameter families of indecomposable modules can be attributed to the algebra automorphism action etc.

In this introductory paper we show give several easy first results and examples for finite dimensional algebras.

2 Definitions

2.0.1 Restriction isomomorphism

Let k be a field, A an associative k -algebra. All modules considered in this paper are left modules. We denote the isomorphism class of a module M by $[M]$. If the algebra A is given by a presentation $\langle X|\rho \rangle$ then the image of X under the algebra homomorphism defining the A -module M is called the set of A -action generators for M . The categories of all and finitely generated A -modules are denoted by $Mod(A)$ and $mod(A)$, respectively. By a subalgebra we mean a subalgebra with the multiplicative identity. Given a subalgebra $A' \subseteq A$ we denote the restriction of M to A' by $res_{A'}(M)$. By e_{ij} we denote the basis matrix having one nonzero entry at position (i, j) .

Definition 2.1 *A -modules M_1 and M_2 are called restriction isomorphic (R -isomorphic, denoted by $[M_1]_R = [M_2]_R$) provided for every proper subalgebra $A' \subset A$*

$$[res_{A'}(M_1)] = [res_{A'}(M_2)] \tag{1}$$

Definition 2.2 *Given a A -module M and a proper subalgebra $A' \subseteq A$ the function*

$$A' \mapsto [res_{A'}(M)]$$

is called the restriction function (total restriction function) of M .

Definition 2.3 *A-module M is called R -decomposable provided $\text{res}_{A'}(M)$ is decomposable for any maximal proper subalgebra $A' \subset A$.*

Definition 2.4 *A-modules M_1 and M_2 are called restriction distinct (R -distinct) provided for every proper subalgebra $A' \subset A$*

$$[\text{res}_{A'}(M_1)] \neq [\text{res}_{A'}(M_2)] \quad (2)$$

2.1 Action of algebra automorphism and twisted isomorphism

Definition 2.5 *A-modules M_1 and M_2 are called twisted isomorphic (T -isomorphic, denoted by $[M_1]_T = [M_2]_T$) provided there exists an A -automorphism $f : A \rightarrow A$ and a k -linear isomorphism (twisted isomorphism or T -isomorphism) $\varphi : M_1 \rightarrow M_2$ such that for every $a \in A$*

$$f(a) \circ \varphi = \varphi \circ a \quad (3)$$

Definition 2.6 *Let $f \in \text{Aut}(A)$, M - a A -module with the list of A -action generators (x_1, \dots, x_n) . $f(M)$ (M twisted by f) is defined as the module with the same underlying space and the action generators $(f(x_1), \dots, f(x_n))$. We call the set $\bigcup_{f \in \text{Aut}(A)} f(M)$ the T -orbit of M .*

We note that T -orbits of A -module isomorphism classes are just equivalence classes under the T -isomorphism equivalence relation and $[M_1]_T = [M_2]_T$ iff $[M_1] = [f(M_2)]$ for some $f \in \text{Aut}(A)$.

Definition 2.7 *We call two A -modules M_1 and M_2 restriction-twisted isomorphic (RT -isomorphic) provided all restrictions to maximal proper subalgebras are T -isomorphic.*

3 Results related to the R -isomorphism

3.1 R -isomorphism is strictly coarser

Proposition 3.1 *R -isomorphism is an equivalence relation on $\text{Obj}(\text{Mod}(A))$ which is coarser than the standard module isomorphism equivalence relation. Both restriction functions are module invariants.*

Sets of module restriction isomorphism types are already being studied in special cases such as modules over modular group algebras restricted to subalgebras isomorphic to modular group algebras for elementary abelian groups of rank 1.

Proposition 3.2 *For algebras of tame and wild representation type R -isomorphism is strictly coarser than the standard module isomorphism equivalence relation.*

Proof We exhibit two counterexamples. For the tame case take $A = k[X, Y]/(X, Y)^2$. Let M_1 be the indecomposable A -module of dimension 3 with $\dim_k(\text{Soc}(M_1)) = 1$ and M_2 be the indecomposable A -module of dimension 3 with $\dim_k(\text{Soc}(M_2)) = 2$. All restrictions for both M_1 and M_2 to maximal proper subalgebras decompose as direct sums of the trivial module and the indecomposable two dimensional module therefore $[M_1]_R = [M_2]_R$. For the wild case let $A = k[X, Y, Z]/(X, Y, Z)^2$. Let M_1 and M_2 be modules of dimension 6 given by (X, Y, Z) action matrix triples (x, y, z) and (x, z, y) , respectively, where

$$\begin{cases} x = e_{41} + e_{52} + e_{63}, \\ y = e_{42}, \\ z = e_{53}. \end{cases} \quad (4)$$

By direct matrix computations it can be proved that $[M_1] \neq [M_2]$ and $[M_1]_R = [M_2]_R$.

3.2 Indecomposable R -decomposable modules

Proposition 3.3 *There exist algebras of wild representation type with indecomposable R -decomposable modules.*

Proof We exhibit an example. Let $A = k[X, Y, Z]/(X, Y, Z)^2$. Let M be the module of dimension 4 given by (X, Y, Z) action matrix triple (x, y, z) where

$$\begin{cases} x = e_{31}, \\ y = e_{32}, \\ z = e_{31} + e_{42}. \end{cases} \quad (5)$$

By direct computation it can be shown that M is indecomposable and R -decomposable.

3.3 R -distinct modules

Proposition 3.4 *There exist algebras of wild representation type with R -distinct modules.*

Proof Let $A = k[X, Y, Z]/(X, Y, Z)^2$. Let M_1 and M_2 be modules of dimension 4 given by (X, Y, Z) action matrix triples (x_i, y_i, z_i) where

$$\begin{cases} x_1 = 0, \\ y_1 = e_{41}, \\ z_1 = e_{31} + e_{42} \end{cases} \quad (6)$$

and

$$\begin{cases} x_2 = e_{41}, \\ y_2 = e_{31} + e_{42}, \\ z_2 = e_{42}. \end{cases} \quad (7)$$

By direct matrix computations it can be shown that M_1 and M_2 are R -distinct.

4 Results related to the T -isomorphism

4.1 T -isomorphism is strictly coarser

Proposition 4.1 *T -isomorphism is an equivalence relation on $\text{Obj}(\text{Mod}(A))$ which is coarser than the standard module isomorphism equivalence relation.*

Proof Equivalence property of the T -isomorphism relation follows from the group properties of $\text{Aut}(A)$. The strictness is proved below in the next subsection.

4.2 Group action property and preservation of indecomposability

Proposition 4.2 *The map $\text{Obj}(\text{mod}(A)) \rightarrow \text{Obj}(\text{mod}(A))$ defined by $M \mapsto f(M)$ is a group action which preserves indecomposability.*

Proof The finitely generated A -modules M and $f(M)$ have the same endomorphism algebras therefore they are simultaneously indecomposable or decomposable.

4.3 Comparing T -isomorphism and R -isomorphism

Proposition 4.3 *T -isomorphism does not imply R -isomorphism and vice versa.*

Proof Counterexamples for the first part of the statement are easy to find for the algebra $A = k[X, Y]/(X, Y)^2$.

To exhibit a counterexample for the second part (R -isomorphism does not imply T -isomorphism) take $A = k[X, Y]/(X, Y, Z)^2$. Let M_1 and M_2 be modules of dimension 6 given by (X, Y, Z) action matrix triples (x_i, y_i, z_i) where

$$\begin{cases} x_1 = e_{51} + e_{42}, \\ y_1 = e_{61} + e_{53}, \\ z_1 = e_{52} + e_{43} + e_{63}. \end{cases} \quad (8)$$

and

$$\begin{cases} x_2 = x_1, \\ y_2 = y_1, \\ z_2 = e_{41} + e_{62} + e_{63}. \end{cases} \quad (9)$$

By direct matrix computations it can be shown that M_1 and M_2 are R -isomorphic but not T -isomorphic.

4.4 One-parameter families of indecomposable modules over tame algebras as T -orbits

We assume that k is algebraically closed and all A -modules considered below are finitely generated. By a family of indecomposable modules we mean a subset of isomorphism classes of $Obj(mod(A))$ continuously depending on arguments taking values in an open subset of k^r for some $r \in \mathbb{N}$, r is said to be the number of parameters of the family. In this section we show that for several classic tame algebras one-parameter families of indecomposable finitely generated modules are T -orbits or their subsets. This also shows that T -isomorphism is strictly coarser than the standard isomorphism.

4.4.1 $A = k[X]$

Modules in one-parameter families of indecomposable finitely generated A -modules are isomorphic to modules with X action matrices being Jordan block matrices. The group $\text{Aut}(A)$ is the group of A -endomaps which fix the element $1 \in A$ and which are obtained by extending the maps of form $X \mapsto aX + b$, $a \neq 0$ according to algebra structure of A . For any $\lambda_1 \in k, \lambda_2 \in k$, $\lambda_1 \neq \lambda_2$ and any $a' \neq 0$ we can find $b' \in k$ such that $\lambda_2 = a'\lambda_1 + b'$. Choosing the A -automorphism f generated by the map $X \mapsto a'X + b'$ and an appropriate diagonal basis change φ which makes off-diagonal elements equal to 1 we see that for any $n \in \mathbb{N}$ we have $[J(\lambda_1, n)]_T = [J(\lambda_2, n)]_T$. Thus we have the following proposition.

Proposition 4.4 *For any $n \in \mathbb{N}$ the one-parameter family $\{J(\lambda, n)\}_{\lambda \in k}$ is a T -orbit.*

4.4.2 $A = k[X, Y]/(X, Y)^2$

Modules in one-parameter families of indecomposable finitely generated A -modules are isomorphic to modules in the families $\{K(\lambda, n)\}_{\lambda \in k \cup \lambda = \infty}$ where modules $K(\lambda, n)$ are given by action generators $K_X(\lambda, n)$ and $K_Y(\lambda, n)$ for X and Y , respectively, which can be chosen as

$$\left\{ \begin{array}{l} K_X(\lambda, n) = \left[\begin{array}{c|c} O_n & O_n \\ \hline E_n & O_n \end{array} \right] \\ , K_Y(\lambda, n) = \left[\begin{array}{c|c} O_n & O_n \\ \hline J(\lambda, n) & O_n \end{array} \right] \end{array} \right. \quad (10)$$

for any $\lambda \in k$ and additionally

$$\left\{ \begin{array}{l} K_X(\infty, n) = \left[\begin{array}{c|c} O_n & O_n \\ \hline J(0, n) & O_n \end{array} \right], \\ K_Y(\infty, n) = \left[\begin{array}{c|c} O_n & O_n \\ \hline E_n & O_n \end{array} \right] \end{array} \right. \quad (11)$$

For more details about finitely generated A -modules see [1]. $\text{Aut}(A) \simeq GL(2, k)$ and is formed by the maps which fix $1 \in A$ and map (X, Y) to $(a_{11}X + a_{12}Y, a_{21}X + a_{22}Y)$ with $\det([a_{ij}]) \neq 0$. For any $\lambda_1 \in k$, $\lambda_2 \in k$, $\lambda_i \neq \infty$, $\lambda_1 \neq \lambda_2$ as in the previous example we

can find $b' \in k$ such that $\lambda_2 = a'\lambda_1 + b'$, $a' \neq 0$. By taking the automorphism given by $(X, Y) \mapsto (X, a'Y + b')$ and a diagonal basis change as in the previous example we see that $[K(\lambda_1, n)]_T = [K(\lambda_2, n)]_T$. By taking $\varphi = id$ and the A -automorphism permuting X and Y we see that $[K(0, n)]_T = [K(\infty, n)]_T$.

Proposition 4.5 *For any $n \in \mathbb{N}$ and any $\lambda \in k$ the one parameter family $\{K(\lambda, n)\}_{\lambda \in K \cup \infty}$ is a T -orbit.*

We end considering this example by noting that A -module isomorphism classes which do not lie in infinite families (the indecomposable A -modules of odd dimension) coincide with T -isomorphism classes.

4.4.3 $A = k[X, Y]/(X^2, Y^2, (XY)^k X^{\epsilon_1}, (YX)^k Y^{\epsilon_2})$

In this case A is closely related to group algebras for dihedral 2-groups over fields of characteristic 2. Modules in one-parameter families of indecomposable finitely generated A -modules are isomorphic to module in the families $\{B(w, \lambda, m)\}_{\lambda \in k, \lambda \neq 0}$ where modules $B(w, \lambda, m)$ are given by action generators $B_X(w, \lambda, m)$ and $B_Y(w, \lambda, m)$ (of size nm) for X and Y , respectively, which can be chosen as

$$B_X(w, \lambda, m) = \underbrace{B_X(w, \lambda) \oplus \dots \oplus B_X(w, \lambda)}_{m \text{ times}}$$

$$B_Y(w, \lambda, m) = \begin{bmatrix} B_Y(w, \lambda) & O_n & O_n & \dots & O_n \\ O_n + e_{nn} & B_Y(w, \lambda) & O_n & \dots & O_n \\ O_n & O_n + e_{nn} & B_Y(w, \lambda) & \dots & O_n \\ \dots & \dots & \dots & \dots & \dots \\ O_n & O_n & O_n & \dots & B_Y(w, \lambda) \end{bmatrix} \quad (12)$$

and the matrices $B_X(w, \lambda), B_Y(w, \lambda)$ of size n are determined by the oriented edge labelled graph isomorphism class of an admissible oriented edge labelled cycle w and parameter $\lambda \in k$ (see [2]). An admissible oriented edge labelled cycle has edges labelled by X and Y , it is subject to conditions determined by the structure of A (see [2]). Note that there is a partition of the vector space basis into groups of n elements corresponding to subquotients

B_1, \dots, B_m each of which is isomorphic to the A -module $B(w, \lambda) = B(w, \lambda, 1)$ given by action matrices $B_X(w, \lambda)$ and $B_Y(w, \lambda)$. Note that $B_X(w, \lambda)$ does not depend on λ and $B_Y(w, \lambda)$ has one of its nonzero entries equal to λ and other nonzero entries equal to 1.

There are A -automorphisms which fix $1 \in A$ and map (X, Y) to (X, aY) , we denote such an A -automorphism by f_a . We want to show that for any one-parameter family $\{B(w, \lambda, m)\}_{\lambda \in k}$ any two distinct elements $B(w, \lambda_1, m)$ and $B(w, \lambda_2, m)$, $\lambda_i \neq 0, \lambda_1 \neq \lambda_2$, are T -isomorphic via a suitable A -automorphism of type f_a . We first show that $[B(w, \lambda_1)]_T = [B(w, \lambda_2)]_T$. We start with module $B(w, \lambda_1)$ and apply an automorphism f_a for some a to it. After this operation the action generator for X does not change, the action generator of Y is multiplied by a . By consecutive diagonal basis changes we can isomorphically transform $f_a(B(w, \lambda_1))$ to a module $B(w, \lambda_1 a^z)$ for some $z \in \mathbb{Z}$. Since k is algebraically closed we can solve the equation $\lambda_1 a^z = \lambda_2$ with respect to a and thus we have that $[B(w, \lambda_1)]_T = [B(w, \lambda_2)]_T$ via the computed A -automorphism f_a . Now suppose we have two modules $B(w, \lambda_1, m)$ and $B(w, \lambda_2, m)$ with $m > 1$. We produce a suitable twisted isomorphism φ in two steps of basis changes. We first transform separately each of the subquotients B_i as described above using diagonal basis changes to have each subquotient isomorphic to $B(w, \lambda_2)$. In the second step we consecutively multiply all basis elements of each subquotient B_i by the same coefficient (which will be different for different subquotients) to make the elements in the off-diagonal blocks for the Y matrix equal to 1. Thus we have the following proposition.

Proposition 4.6 *For any $m \in \mathbb{N}$ and any admissible word w the one-parameter family $\{B(w, \lambda, m)\}_{\lambda \in k, \lambda \neq 0}$ is a subset of a T -orbit.*

4.5 One parameter family does not necessarily belong to one T -orbit

Let $A = k[x, y]/(x^2, y^3, y^2 - (xy)x)$. If $\text{char}(k) = 2$ then A is a semidihedral group algebra.

Proposition 4.7 *There are one parameter families of indecomposable A -modules which do not belong to one T -orbit.*

Proof We will produce an example. Let $f : A \rightarrow A$ be an A -automorphism, $f(x) = \tilde{x}$, $f(y) = \tilde{y}$. We must have

$$\begin{cases} \tilde{x}^2 = 0, \\ \tilde{y}^3 = 0, \\ \tilde{y}^2 = \tilde{x}\tilde{y}\tilde{x}. \end{cases}$$

Let

$$\tilde{x} = a_1x + a_2y + a_3xy + a_4yx + a_5xyx + a_6yxy.$$

Relation $\tilde{x}^2 = 0$ implies $a_2 = 0$. For f to be bijective we must have $a_1 \neq 0$.

Let M_1 and M_2 be modules of dimension 2 given by (X, Y) action matrix pairs (x_i, y_i) where

$$\begin{cases} x_1 = e_{21}, \\ y_1 = 0 \end{cases} \quad (13)$$

and

$$\begin{cases} x_2 = 0, \\ y_2 = e_{21}. \end{cases} \quad (14)$$

M_1 and M_2 belong to a one parameter family of indecomposable A -modules defined by matrix pairs $\{(e_{21}, \lambda e_{21})\} \cup \{(0, e_{21})\}$.

If M_1 and M_2 would belong to one T -orbit then there would exist an A -automorphism f such that $f(x_1) = x_2 = 0$. This is not possible since f coefficient at x is not 0.

4.6 Examples of multiparameter module families over a wild algebra as a subset of a T -orbit

4.6.1 A 2-parameter family

Let $A = k[X, Y, Z]/(X, Y, Z)^2$. A is known to be a minimal wild algebra, i.e. all its subalgebras are of tame or finite representation type. Clearly $\text{Aut}(A) \simeq GL(3, k)$ and is formed by maps which fix $1 \in A$ and map (X, Y, Z) to $(a_{11}X + a_{12}Y + a_{13}Z, a_{21}X + a_{22}Y + a_{23}Z, a_{31}X + a_{32}Y + a_{33}Z)$ with $\det(a_{ij}) \neq 0$. Consider a family $\mathcal{C}_2 = \{C(\alpha, \beta)\}$, $\alpha \in k, \beta \in k, \alpha \neq 0, \beta \neq 0$, of nonisomorphic indecomposable A -modules of dimension 2 given by action generators

$$\left\{ \begin{array}{l} C_X(\alpha, \beta) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \\ C_Y(\alpha, \beta) = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}, \\ C_Z(\alpha, \beta) = \begin{bmatrix} 0 & 0 \\ \beta & 0 \end{bmatrix} \end{array} \right., \quad (15)$$

By considering diagonal A -automorphisms of form $(X, Y, Z) \rightarrow (X, aY, bZ)$ we see that $[C(\alpha, \beta)]_T = [C(1, 1)]_T$ and thus the family \mathcal{C}_2 is a subset of a T -orbit.

4.6.2 A 3-parameter family

Let $A = k[X, Y, Z]/(X, Y, Z)^2$. Consider a family $\mathcal{C}_3 = \{C(\alpha, \beta, \gamma)\}$, $\alpha \in k, \beta \in k, \alpha \neq 0, \beta \neq 0, \gamma \neq 0$, of nonisomorphic indecomposable A -modules of dimension 5 given by action generators

$$\left\{ \begin{array}{l} x(\alpha, \beta) = e_{41} + e_{32}, \\ y(\alpha, \beta) = e_{51} + e_{42}, \\ z(\alpha, \beta) = \alpha e_{32} + \beta e_{41} + \gamma(e_{51} + e_{42}). \end{array} \right. \quad (16)$$

By direct computation it can be shown that \mathcal{C}_3 is a subset of a T -orbit.

5 Research directions

In this subsection we list several research problems related to the described coarsifications of the isomorphism.

5.1 Multiparameter families of indecomposable modules

1. Do there exist multiparameter families of indecomposable nonisomorphic and X -isomorphic modules ($X \in \{R, T, RT\}$)? Describe such families.
2. Do there exist multiparameter families of indecomposable nonisomorphic modules which are transversal with respect to the X -isomorphism orbits (all elements are pairwise non- X -isomorphic)? Describe such families.

5.2 Restriction functions

1. What are the admissible restriction functions?
2. What are the sets of nonisomorphic modules for each value of the restriction function?
Describe the distribution. In particular, for which values of the restriction function the sets of nonisomorphic modules are wild (tame, finite)?
3. Describe restriction functions of modules in almost split sequences.
4. Describe R -decomposable modules.

5.3 T -orbits

1. What are the admissible T -orbits?
2. How do T -orbits depends on module structure of its representatives?
3. Describe multiparameter families of indecomposable modules which belong (do not belong) to a single T -orbit.

References

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